Petr Vopěnka

New Infinitary Mathematics

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Editor's Note

The original reason for this book was the consensus that Vopěnka's mathematical and philosophical contributions made after he left mainstream set theory should be available in English. Bringing the book to publication has taken ten years for the following reasons: first Vopěnka wrote another manuscript in Czech[1] subsequently translated by Hana Moravcová and Roland Andrew Letham, called *The Great Illusion of the Twentieth Century Mathematics*. However, it turned out that the translation of some parts of the text needed more relevant mathematical expertise and Alena Vencovská took on the task of making it correct. The author used the opportunity to extend and modify the book considerably. He worked on it until his sudden death in 2015. The result was twofold: more publications in Czech, namely the four-volumed work *New Infinitary Mathematics,*[2] along with *Prolegomena to the New Infinitary Mathematics,*[3] and a parallel English text with additions to the original book translated by Vencovská. The Czech and English versions differed little from each other, except that the order of the material was different, and Vopěnka left some parts out from the English version. In particular, he did not include what are now the first two chapters, and some sections throughout. This present version does include these initial chapters (on the theological foundations of Cantor's set theory and on its rise and growth, the former translated by Václav Paris) but it does not include all that is in the Czech version.

[1] Petr Vopěnka, *Velká iluze Matematiky XX. stoleti a nové základy* (Plzeň: Západočeská univerzita v Plzni a Nakladatelství Koniáš, 2011).

[2] Petr Vopěnka, *Nová infinitní matematika* (Praha: Karolinum, 2015).

[3] Petr Vopěnka, *Prolegomena k nové infinitní matematice* (Praha: Karolinum, 2013).

Editor's Introduction

About the Author

Petr Vopěnka grew up in the former Czechoslovakia, where he was born in 1935 (to parents who both taught mathematics at a secondary school). He enjoyed scouting in his youth and often remembered times spent at camps. In a way he remained true to the values he formed early on all through his life. Personal integrity, faith in truth prevailing over deceit, loyalty to friends, great love for his troubled country and an unshakeable commitment to his work were some of his most striking characteristics. To this, one needs to add that he loved to laugh.

For much of his life, Czechoslovakia was ruled by the communists: they took over in 1948, and education during Vopěnka's teenage years bore the stamp of Stalinism. Vopěnka reminisced about being asked to take turns in a whole day of reading funereal poems on the school radio upon Stalin's death in 1953, and he arrived in Prague later the same year to study mathematics in a city overlooked from a hill by Stalin's 16-meter-high statue. Fortunately, mathematics is relatively immune to ideological manipulation and Vopěnka remembered his student years and his teachers fondly.

His early research was mainly in topology and he wrote his master's thesis under the supervision of Eduard Čech, an eminent topologist and geometer, whose name lives for example in Čech cohomology and Čech-Stone compactification. Vopěnka used to say that Čech "showed him how to do mathematics". The research that he engaged in at that time concerned compact Hausdorff spaces and their dimensions.

Soon after graduation, Vopěnka started to teach mathematics at Charles University and he remained there for most of his professional life. Quite early on, he developed an interest in mathematical logic, championed in Czechoslovakia by Ladislav Rieger who wrote about the subject for the Czech mathematical community and ran a seminar on set theory. Vopěnka participated and, after Rieger's untimely death in 1962, took over as its organiser to provide strong and inspired leadership for Czechoslovak mathematical logicians. Vopěnka published work on nonstandard interpretations of Gödel-Bernays set theory based on using the ultrapower construction and then in collaboration with the seminar participants he contributed substantially to the exciting discoveries following Gödel and Cohen's groundbreaking work on the consistency and independence of the continuum hypothesis and the axiom of choice. Due to the Iron Curtain, communication with other mathematicians working in the area was limited and some results obtained independently in Prague came later than those in the West, but others remain credited to the Prague group. By all accounts it was as vibrant and fruitful a period as can be – Alfred Tarski wrote about the community in these words:[4]: "I do not know if there is at this point another place in the world, having as numerous and cooperative a group

[4] Quoted in Antonín Sochor, "Petr Vopěnka (born 16. 5. 1935)," *Ann. Pure Appl. Logic* 109 (2001): 1–8.

of young and talented researchers in the foundations of mathematics."

This lasted some years, but then two factors caused it to fall apart. One was that Vopěnka became very sceptical about the role that set theory, as it was, could have in truly explaining the phenomenon of infinity and in serving as a foundation for mathematics. It mattered to him; he did not wish to explore that intricate and bewitching maze any further so he started to look for alternatives. Paradoxically, the one concept which is today perhaps most strongly associated with Vopěnka within this area arose as he was abandoning the subject, when he proposed what became to be known as Vopěnka's principle. This yields a strong large-cardinal axiom that Vopěnka said he believed he could prove to be contradictory, suggesting it merely to make the point that investigating consequences of more and more set-theoretical axioms made little sense. However, Vopěnka's argument that it was contradictory contained an error, and interest in the axiom prospered outside of Czechoslovakia. Tightening controls within the country again limited communication with the West for academics like Vopěnka so it was some years later that he learnt with surprise that this principle was still alive and well established.

The other factor that contributed to the demise of this golden era of mainstream set theory in Prague were the political events – the 1960s brought a gradual thaw of orthodox communism leading to Prague Spring in 1968. This however was followed by the August 1968 invasion whereby the Warsaw Pact armies put an end to it. Some of Vopěnka's collaborators, in particular Tomáš Jech and Karel Hrbáček left the country, and most of the others sought their own independent paths. Vopěnka, who prior to 1968 had joined the efforts led by Alexander Dubček to reform communism and had gained some influence in running the Faculty of Mathematics and Physics at his university, would not support the official line after the invasion and might well have been forced to leave the university along with many other academics in similar positions. He was allowed to stay to do research, although his contact with students was very restricted. Many years later when he learnt that he owed this good fortune to the intervention of the Soviet mathematician P. S. Alexandrov, he used to joke that had he known how powerful a protector he had, he would have been braver (standing up to the suffocating pressure of the Czechoslovak communist "normalisation" of the 1970s and 1980s). In fact, he was one of the few who did stand up to it in any way that seemed possible.

At this turning point, Petr Vopěnka along with Petr Hájek wrote a book on semisets,[5] exploring set theories obtained by modifying the usual von Neumann-Bernays-Gödel axioms for classes and sets so that sets can have subclasses that are not themselves sets (proper semisets). Apart from the importance of semisets for forcing, Vopěnka's new motivation was investigating other ways in which the phenomenon of infinity could be captured mathematically, better reflecting how we encounter infinity when thinking about the world, often as a part of a large finite set. The book did not dwell on this aspect though and

[5] Petr Vopěnka and Petr Hájek, *The Theory of Semisets* (Prague: North Holland and Academia, 1972).

focused on providing a careful formal development of the theory of semisets and on showing its suitability for finding models of set theory via forcing.

Vopěnka then moved on to formulate a different set theory, which he hoped would capture his intuition about infinity in a better way. It was an intuition gained through much reflection on what we understand by infinity and how we see the world, influenced mainly by Bolzano and Cantor's writings, by discussions surrounding the birth of set theory and by the philosophy of Husserl and Heidegger (for many years there was a weekly seminar taking place in Vopěnka's office devoted to the study of their work). Semisets were a step in the right direction, but Vopěnka wished to formulate a new theory from the position of a mathematician free of any commitment to the current view of infinity; to develop mathematics as it might have been developed if satisfactory axioms for infinitesimals had been found before mathematics took its present course.

This led to what he called the alternative set theory. It contains sets and classes; sets alone behave as classical finite sets but they may contain subclasses which are not sets (semisets). Unlike Cantorian set theories, alternative set theory admits only two types of infinity: the countable infinite and the continuum. This is not a necessary requirement of such a set theory, it could be constructed otherwise, but Vopěnka's motivation was to keep only what could be justified by some intuition other than intuition arising purely from the study of set theory itself; for him it meant just the infinities associated with natural numbers or with the real line. A crucial principle in Vopěnka's alternative set theory is the Axiom of Prolongation, related to the phenomenon of the horizon (understood in a very general sense). It reflects the intuition that something seen to behave in a certain well-defined way as far as the horizon will continue to do so beyond the horizon.

Mathematically, the theory is close to the concept of nonstandard models of natural numbers underlying nonstandard analysis. However, from a foundational point of view there is a considerable difference since in nonstandard analysis infinitesimals are complicated infinitary objects whilst in AST some exist just as rational numbers do. Formulating a theory that allows mathematical analysis to be practiced in a way in which it was conceived by Leibniz, that is as a calculus with infinitesimals, was indeed one of Vopěnka's objectives. This had not been done within the alternative set theory at the time, and Vopěnka returned to the task in this book.

Vopěnka succeeded in assembling another group of enthusiastic mathematicians, who wanted to work with him and develop AST. One unfailingly supportive and faithful collaborator from before also joined him in the endeavour, Antonín Sochor. Interesting results were obtained, first within the Prague circle and later on also at other places in the world, but overall its impact was relatively small. In particular, investigations of alternative set-theoretical universes was restricted to what Vopěnka called a limit universe (as opposed to a witnessed universe). In a limit universe no "concrete" set such as the set of natural numbers less than $67^{293^{159}}$ can contain semisets but in a witnessed universe some can. The witnessed universes correspond to Vopěnka's intuition, but their theory is classically inconsistent (Vopěnka envisaged some approach involving the convincingness of proofs).

The first comprehensive account of AST appeared in 1979 in a monograph by Vopěnka.[6] In 1980 there was supposed to be a Logic Colloquium in Prague where AST would surely have been widely discussed and whatever stand logicians would have taken, its ambition to lead to new foundations for mathematics would have attracted more attention. However, shortly before the Colloquium was due to start the communist regime revoked the permission for it to take place, because the logic community was calling for the release of an imprisoned Czech logician and the regime feared the negative publicity. The next Logic Colloquium in Prague had to wait eighteen years, nine years after the Velvet Revolution. Vopěnka was an honorary chairman and his opening words are very telling, both of the man and the bygone times:

> "Ladies and Gentelmen, I am very happy to be able to welcome you to Prague. French historian Ernst Denis once wrote that in Prague every stone tells a story. As you walk across the Charles Bridge, pause to remember Tycho Brahe and Johannes Kepler who used to stroll there over 400 years ago as well as Bernard Bolzano two centuries later. I am sure that you too will fall in love with this old, inspiring, majestic, but also tragic city."
>
> These were the words with which I had planned to welcome participants of Logic Colloquium '80 which was cancelled by the communist government. The totalitarian regime was afraid that the participating mathematicians would call for the release of their colleague, mathematician Vaclav Benda, who was serving a five year prison term. He was imprisoned for publicly drawing attention to politically motivated prosecution of those opposing the regime. For us, Czech mathematicians, the cancellation meant even deeper isolation from our colleagues abroad. But we never doubted that even though mathematics is very beautiful, freedom is even more so. Logic Colloquium '98 will now commence.

By this time Vopěnka had entered yet another stage in his professional life. After the demise of communism in 1989 he had served as the Minister of Education in the new democratic government, throwing all his passion and energy into trying to reform the education system, with mixed success. After completing his term of office, he returned to academia but devoted himself mainly to the history and philosophy of mathematics. He wrote several books, in Czech, most notably *The Corner Stone of European Learning and Power* (Úhelný kámen evropské vzdělanosti a moci, 1998), *Narration about the Beauty of Neo-baroque Mathematics* (Vyprávění o kráse novobarokní matematiky, 2004) and *Meditations on the Foundation of Science* (Meditace o základech vědy, 2001). In 2004 he was awarded the Vize 97 prize by The Dagmar and Vaclav Havel Foundation designated by the charismatic Czech playwright president for "significant

[6] Petr Vopěnka, *Mathematics in the Alternative Set Theory* (Leipzig: Teubner, 1979).

thinkers whose work exceeds the traditional framework of scientific knowledge, contributes to the understanding of science as an integral part of general culture and is concerned with unconventional ways of asking fundamental questions about cognition, being and human existence." Vopěnka continued his work in the same spirit, eventually returning again to mathematics to describe his stand on its foundations.

About the Book

There are some features in Vopěnka's work which it is useful to highlight. Vopěnka wrote extensively (in Czech) about ancient Greek geometry and its development throughout the centuries and about the origins and assumptions of set theory. It was essential for him to understand what mathematicians were doing, and he always wanted to see beyond the formal side of it: proving theorems from axioms did not suffice. He needed to know why anything should be assumed and this led him to formulate his own philosophical standpoint and develop his own terminology.

This is particularly important for his arguments about sets, which he discusses in this book. He explained his positions in detail for example in his book *Meditations on the Foundations of Science*.[7] It was influenced by the philosophy of Edmund Husserl and his followers but Vopěnka adapted the phenomenological program in his own way. The starting point are phenomena we encounter; from those we create objects by conceding them a "personality". It does not matter what is the character of the phenomenon in question, it could be something we perceive or remember or just think. When we single out some objects from those previously created, we can collect them together and when we consider them thus collected and without their various properties and interrelations, we make a collection of objects. Thus collections are determined exclusively by the presence of the objects belonging to them: belonging is not graded, and an object either belongs or not. When we consider a collection as an object, that is concede a personality to it, we make it into a class. The difference is that a collection is a multiplicity of objects but a class is a single object. As an object, it can belong to other collections. A class is uniquely determined by its members and, conversely, it uniquely determines the collection of its members although this can be in various ways and it may not be possible simply to list the members. A set is a class such that the collection of its members is sharply defined. For non-sharply defined collections, Vopěnka refers to examples like the numbers of grains taken from a heap of sand that still leave a heap. A semiset is defined to be a class which is a subclass of some set but not itself a set (where a class X is a subclass of a class Y if all members of X belong to Y).

Apart from collections, Vopěnka uses the notion of domains. He writes

> When talking about people, we often think not only of people who are alive at that moment or have lived in the past, but also of those who are yet to be born or even of those who have never been born nor

[7] Petr Vopěnka, *Meditace o základech vědy* (Prague: Práh, 2001).

> ever will be. The extension of the concept of people is therefore not a collection, but a domain of all people. A domain is not a totality of existing objects (regardless of the modality of their existence); it is the source and simultaneously also a sort of container into which the suitable emerging or created objects fall. Naturally, every collection of objects can be interpreted as a domain, albeit an exhausted one. By actualising a domain we mean exhausting the domain, that is, substituting this domain by a collection of all the objects that fall or can fall into it.[8]

Thus it is some way from a domain to a set, and the questions of whether a domain can be actualised and whether this would yield a set is of fundamental importance.

In *Meditations*, Vopěnka gives an explanation of abstract objects and then he says:

> Abstract objects are the building blocks of the remarkable world of abstract mathematics. The modality of their being is some special, separated (abstract), and yet changeable being. These phenomena arise from nothingness by the strength of our will and their being culminates when they are captured in our minds. If we stop thinking them, they do not perish; just the modality of their being decreases. As if the nothingness slowly absorbed these phenomena but was no longer able to absorb them completely. Hence it at least hides them under the ever-condensing cover of emptiness from which they again surface when we remember them. We will refer to this idiosyncratic being of abstract objects as existence.[9]

It is in this light that we need to understand his arguments about Cantor's set theory and about the existence of the set of natural numbers. (By Cantor's set theory Vopěnka means any considerations based on Cantor's ideas, be it within the most commonly used ZFC – Zermelo Fraenkel set theory with the axiom of choice, or GB – Gödel Bernays set theory, within which he himself worked in the 1960s, or some other system based on the same approach to infinity.) For abstract objects with certain properties to exist, it must be possible to think them so at the very least there cannot be an apparent contradiction in them. But that is not all: we as finite beings should not really be able to think beyond the finite. So what is it that gives us the confidence to do so?

Vopěnka went further back, and started by asking how Euclidean geometry was possible. He argued that the mathematics of the ancient Greek world, that is, the way in which people thought about it, appropriated the capabilities of the Olympian gods to grasp the unchanging truth in the changing world. He refers to Zeus, or to a superhuman, as the performer of ancient (Greek) geometry. Zeus can extend a straight line further than any limit we may come up with

[8] Petr Vopěnka, *Prolegomena k nové infinitní matematice* (Prague: Karolinum, 2015).
[9] Petr Vopěnka, *Meditace o základech vědy* (Prague: Práh, 2001).

and he can see how a straight line approaches to touch a circle. Still, he does not wield absolute power and he could not hold in his mind all that there is. Such power does however belong to the God of medieval scholastic philosophy and using it made Cantor's set theory possible. It was in fact Bolzano, half a century before Cantor developed his theory of infinity, who came up with a proof of the existence of an infinite set (the only proof ever given, as Vopěnka used to say). Bolzano's proof is discussed in Section 1.3. Accordingly, Vopěnka sometimes refers to God, or to a God-man,[10] as the performer of the classical (modern) geometry and mathematics, on the grounds of it being based on Cantor's set theory. Faced with the question of how to perform mathematics now, Vopěnka notes that in the twenty first century, theological support is no longer there and he proposes his New Infinitary Mathematics, in the spirit of the alternative set theory.

The book has the following structure: Part I is a historical, philosophical and mathematical introduction. The author discusses the history of approaches to infinity up to the time when actually infinite sets became an integral part of mathematics. He shows how fundamental a role theological considerations played in enabling Bolzano and Cantor to produce work that established actually infinite sets as a legitimate object of study. Then he outlines the development of the basic ideas of set theory, focusing on the intuition that guided those early pioneers of set theory before the axiomatic frameworks found their final forms. He argues informally, attempting to capture the spirit of what appeared in the early days as the best way to build set theory; this includes the Axiom of Choice. Finally he argues that stripped of the support of medieval rational theology, we lose more than just certainty that actually infinite sets exist. To wit, assuming the actual existence of the set of all natural numbers (identified with their von Neumann's representations) leads, via the ultrapower construction and the ultraextension operator, to another set of all natural numbers containing all the previous ones and more, which is absurd. Although only some of the obvious questions and objections to this argument are answered in Vopěnka's text, one of his aims was to provoke a debate, and there is much that can be said. Part II proposes a new framework for mathematics while carefully motivating why it should be built in this way. The crucial concepts are those of natural real world, natural infinity and horizon. Mathematically, it is similar to the alternative set theory although there are differences, for example nothing corresponds to the axiom of two cardinalities which is adopted therein. Vopěnka saw his theory as an open challenge to be developed further; in particular he felt that predicate calculus may not be the only tool with which to study it. However, he did not investigate this further. In part III the author seeks to provide rigorous foundations for the development of the infinitesimal calculus on the basis of his theory. This is similar to Abraham Robinson's treatment of calculus in non-standard analysis, but Vopěnka's aim is to resurrect the original intuition that guided Leibniz, and to work with infinitesimals that actually exist as finite objects, without the need for them to be representatives of other, infinitary ob-

[10] See page 215.

jects. Part IV is shorter than the others and it is devoted to the real numbers. The four parts mirror the four volumes of the Czech version of *New Infinitary Mathematics*, with the first part including also some of *Prolegomena*. The circumstance that Vopěnka was simultaneously preparing this book and the Czech version[11] should explain some repetitions and variations in the present volume, although effort has been made to minimise them.

Regarding Vopěnka's style, it is useful to note that he frequently specifies the default meaning of symbols or letters at the start of various chapters or sections to be valid within those chapters or sections (or even during a section, to be valid until the end) and he does not necessarily repeat this when the symbols are used in theorems etc.

In the process of arranging for the publication of this work in English, some serious objections were raised, most notably the failure of the author to engage with the more recent scientific and philosophical literature and relate his thoughts to it. This is justified and could be damning, but there is much to redeem the book. It is a serious attempt by a leading mathematician to re-work the foundations of mathematics at a time when many mathematicians prefer to divorce their subject from the obligation to understand its own foundations. Cantorian set theory has in general been taken to provide such a foundation but the fact that there appears to be no one *true* classical set theoretical universe has made this hard to uphold. In a recent article[12] Akihiro Kanamori writes:

> Stepping back to gaze at modern set theory, the thrust of mathematical research should deflate various possible metaphysical appropriations with an onrush of new models, hypotheses, and results. Shedding much of its foundational burden, set theory has become an intriguing field of mathematics where formalized versions of truth and consistency have become matters for manipulation as in algebra. As a study couched in well-foundedness ZFC together with the spectrum of large cardinals serves as a court of adjudication, in terms of relative consistency, for mathematical statements that can be informatively contextualized in set theory by letting their variables range over the set-theoretic universe. Thus, set theory is more of an open-ended framework for mathematics rather than an elucidating foundation.

Still, some mathematicians and certainly philosophers of mathematics worry about the truth. Interesting as it would be, this book does not engage in a discussion of how it relates to such literature. Rather, it tries to find the truth from the position of a mathematician in the early 21st century, who spent a lifetime thinking about foundational issues, who is aware of the big metaphysical/theological assumptions behind the current framework and who searches for what is left when we give them up, relying just on human intuition and ability to make sense of the world.

[11] See page xi.

[12] Akihiro Kanamori, "Set Theory from Cantor to Cohen," in *Handbook of the Philosophy of Science; Philosophy of Mathematics,* ed. Andrew Irvine (Elsevier, 2007).

Part I

Great Illusion of Twentieth Century Mathematics

… the boy began to delight in his daring flight, and abandoning his guide, drawn by desire for the heavens, soared higher. His nearness to the devouring sun softened the fragrant wax that held the wings: and the wax melted: he flailed with bare arms, but losing his oar-like wings, could not ride the air…

Ovid, *Metamorphoses* VIII, 185–235.

Chapter 1

Theological Foundations

1.1 Potential and Actual Infinity

The notion of infinity came to be narrowed and made more precise already around the time of the first clashes between Christian theology and ancient science. During that period, the meanings of this term that refer to indefiniteness, elusiveness and uncertainty were suppressed. Later, modern science – along with the influential parts of philosophy and theology associated with it – rid the notion of infinity of such meanings completely. Therefore, we too (unless otherwise stated) will in this chapter give the name infinity only to infinity with a classical interpretation; that is, the infinity that is still associated with this name in scientific circles.

Our primary encounter with infinity interpreted in this way occurs when dealing with sharply defined infinite events, that is, when drawing on some constant, precisely determined, but inexhaustible possibilities. This happens when we repeatedly add one to a number, when we repeatedly extend a line by a given length in classical geometric space, and so forth. This form of infinity has been given the name **potential infinity**. This is because infinity has an influence on such events, it seizes them, and they submit to it. On the other hand, by **actual infinity** we understand the form of the phenomenon of infinity shown in a work created by the exhaustion of all relevant inexhaustible possibilities. For example, if we say that the sequence of natural numbers 1, 2, 3, ... is potentially infinite, then we mean that only finitely many of these numbers are created at any given point, but it is always possible to create more. But if we say that this sequence is actually infinite, then we mean that all natural numbers have already been created (and so no more can be created). Similarly the aforementioned repeated line extension is potentially infinite; the work that emerges after all these possible extensions is no longer a segment, but a straight line (or a half-line), in which the infinity governing this work shows in its victorious, actual form.

Already in the very definition of actual infinity there is an almost too patent contradiction. It demands that what is inexhaustible be exhausted, that the infinite come to an end. That is, what is infinite should in a sense become also finite. Nevertheless, actual infinity was not universally rejected as a logically contradictory concept unacceptable by pure reason. From time to time it crept into the deliberations of certain thinkers.

Modern European science did not, of course, flood with over-confidence in actual infinity. As far as it could, it circumvented it – more or less successfully – by means of potential infinity, whose classical interpretation was universally accepted.

The problem of actual infinity however did come up almost constantly with insistent urgency in connection with interpretations of the Christian God. The problem of actual infinity was thus – and in fact it still is so today – primarily a theological problem, even though since the middle of the nineteenth century modern mathematics has appropriated, modified, and absorbed into itself the part of theology that developed around it. But before this happened, European thinking struggled very hard with this problem. We will look at some important milestones of this struggle in the chapter that we are currently opening.

1.1.1 Aurelius Augustinus (354–430)

Among the thinkers who elevated the Christian God to dizzying heights above all the grandest pagan gods, St. Augustine occupies the leading place. He was the one who, for the greatness and glory of the resplendent majesty of God, looked into the bottomless depths of absolute infinity and decided the battle between the Christian God and this all-consuming and all-creation-defeating depth in God's favour. He did so above all in the eighteenth chapter of the twelfth book of his famous work *De civitate Dei*, bearing the telling title "Against Those Who Assert that Things that are Infinite Cannot Be Comprehended by the Knowledge of God." For illustration, we offer the following excerpt from Marcus Dods' 1887 translation[1] with the simple note that in Augustine's terms, number usually means the count of natural numbers from one to a given natural number.

> As for their other assertion, that God's knowledge cannot comprehend things infinite, it only remains for them to affirm, in order that they may sound the depths of their impiety, that God does not know all numbers. For it is very certain that they are infinite; since, no matter of what number you suppose an end to be made, this number can be, I will not say, increased by the addition of one more, but however great it be, and however vast be the multitude of which it is the rational and scientific expression, it can still be not only doubled, but even multiplied. Moreover, each number is so defined by its own properties, that no two numbers are equal. They are therefore both unequal and different from one another; and while they are simply finite, collectively they are infinite. Does God, therefore, not know

[1] Augustine, *The city of God,* transl. Marcus Dods (Buffalo NY: Christian Literature Publishing, 1887).

> numbers on account of this infinity; and does His knowledge extend only to a certain height in numbers, while of the rest He is ignorant?

1.1.2 Thomas Aquinas (1225–1274)

The great Church teacher St. Thomas Aquinas dealt with the relationship between God and infinity predominantly in connection with God's power. This theologian, perhaps the most influential theologian, subjected the power of God to the necessities of reason and hence, without being willing to admit it, subordinated God to reason. This organic involvement of God in the humanly-sought and reason-ruled order of the real world significantly stimulated and nurtured the development of modern European natural science and science in general, yet at the same time it also surrendered God to it.

St. Thomas Aquinas justified the restriction of God's power on account of God having only active possibilities, and therefore active power (being omnipotent in this sense), and not the possibilities of passiveness, and therefore the power of passivity, for God is pure being. The passive possibility of something is – somewhat loosely speaking – its being something that it could be, but by its very nature it can no longer be. For example, if a bird takes off, then it had the option not to take off, but if it has already taken off, then it has lost that opportunity; it is no longer an active or feasible possibility, but only a passive one.

In the article entitled "Whether God Can Do Not to Be Past," found in the first volume of the *Summa Theologica*, St. Thomas Aquinas says quite openly:

> ...there does not fall under the scope of God's omnipotence anything that implies a contradiction. Now that the past should not have been implies a contradiction. For as it implies a contradiction to say that Socrates is sitting, and is not sitting, so does it to say that he sat, and did not sit. [...] Whence, that the past should not have been, does not come under the scope of divine power. [...] Thus, it is more impossible than the raising of the dead; in which there is nothing contradictory, because this is reckoned impossible in reference to some power, that is to say, some natural power; for such impossible things do come beneath the scope of divine power.[2]

In short, God's power over the real world, that is, over created beings and ideas imprinted onto this world, is limited by the law of logical contradiction and only by it. Thus, the limits of God's power are only the necessities of pure reason.

At a time when the whole world was inside a crystal spherical surface, it was not difficult to gain the conviction that in such a world, actual infinity does not occur. Similarly, it is certain that no right-minded person would object to the fact that no created being is able to see actual infinity. The expected objection is that actual infinity can be discovered at least among the ideal phenomena, and

[2] Thomas Aquinas, *Summa Theologica* 1.1–26.

namely in the mathematical sciences, since geometers are in the habit of saying "let's take this infinite line." St. Thomas Aquinas dismisses this objection in the first part of the *Summa Theologica* as follows:

> A geometrician does not need to assume a line actually infinite, but takes some actually finite line, from which he subtracts whatever he finds necessary; which line he calls infinite.

In other words, geometry does not need to get mixed up in God's preserve, which is actual infinity; it has no reason to do so. After all, no geometrician can see that far.

On the other hand, especially in the cultivation of mathematics, we often encounter infinity, even if only in its potential form. Thomas Aquinas briefly mentions this in picking apart the seventh question of the third volume of the *Summa Theologica*, when he writes:

> If we speak of mathematical quantity, addition can be made to any finite quantity, since there is nothing on the part of finite quantity which is repugnant to addition. But if we speak of natural quantity, there may be repugnance on the part of the form... And hence to the quantity of the whole there can be no addition. [He means, apparently, the sky – that spherical crystal surface].

With these words St. Thomas Aquinas indirectly suggests that it is in the cultivation of the mathematical sciences that we can, through reason, touch – but only timidly – the greatness of God. And also the other way round, as Boethius (480–524) claims, knowledge of things Divine cannot be acquired by anyone completely devoid of mathematical training.

By depriving the real world of actual infinity, granting the right to handle it only to God and denying it to created beings, St. Thomas Aquinas greatly simplified the problem of actual infinity. He transferred it to the exclusive competence of God and so opened up the search for its solutions in a realm beyond human intellectual cognition. Consequently the only question was whether God could really know actual infinity, that is, to show how he knows it in a way that eschews the contradiction contained in the very concept of actual infinity, which appears in it when we approach it from potential infinity. In other words, it is necessary to show how the supreme Christian God overcomes actual infinity. To this end, it helps to be aware of the differences between the intellectual cognitive capacities of God and man. These are pointed out by St. Thomas Aquinas in the first book of the *Summa Contra Gentiles,*[3] where he writes that:

> our intellect does not know the infinite, as does the divine intellect. For our intellect is distinguished from the divine intellect on four points which bring about this difference. The first point is that our intellect is absolutely finite whereas the divine intellect is infinite.

[3] Thomas Aquinas, *Summa Contra Gentiles* 1.69.14.

> The second point is that our intellect knows diverse things through diverse species. This means that it does not extend to infinite things through one act of knowledge as does the divine intellect. The third point follows from the second. Since our intellect knows diverse things through diverse species it cannot know many things at one and the same time. Hence, it can know infinite things only successively by numbering them. This is not the case with the divine intellect which sees many things together as grasped through one species. The fourth point is that the divine intellect knows both the things that are and the things that are not, as has been shown [in one of the earlier chapters of the *Summa contra Gentiles*].

1.1.3 Giordano Bruno (1548–1600)

Three dialogues about actual infinity in the real world are contained in the slim book *De l'infinito universo et Mondi* (*The Infinite Universe and the World*), published in Venice in 1584 by Giordano Bruno. In them, this educated Dominican gradually reveals weaknesses in the teachings of his famous predecessor and confrer, St. Thomas Aquinas (already a saint at that time and declared a church teacher) about the fact that there can be no actual infinity in created things. Aware that Thomas's explanation limits the expressions of God's power and thereby also God's power itself, Giordano Bruno's book opens the way for actual infinity to enter even into the material component of the real world.

Thomas's evidence of the impossibility of actual infinity in the real world is taken almost literally from Aristotle and that is why Giordano Bruno attacks Aristotle. He prefers not to mention Thomas at all out of caution. Yet, this does him no good.

Unlike St. Thomas Aquinas, Giordano Bruno has already, with full consciousness, taken the fateful step of modern science, which consists in inserting the real world into classical geometric space. This step was so captivating that until the time of Riemannian geometry, that is, until the middle of the nineteenth century, it was irreversible.

According to Aristotle, real space extends all the way to that spherical crystal surface dotted with fixed stars. There is no place behind it. This spherical surface is the boundary of the real world and is its place. If it did not exist, there would be no place left after it.

Such a ridiculously small piece of work as would be the real world of Aristotle and St. Thomas Aquinas, however, not only would not correspond to God's infinite goodness, but it would also not reflect God's infinite majesty, and most importantly – it would not be worthy of God's infinite power, which in its unattainable magnitude can in fact be manifested only by an actually infinite deed. These reasons – which point in favour of an actually infinite number of different suns and planets in the universe, many of which appear as stars in the night sky – are noted by Giordano Bruno in the following words:[4]

[4] Rather than offering a literal translation of Bruno's Italian dialogues, the English translations presented here seek to preserve the interpretation of J. B. Kozák's Czech edition, used

> Infinite majesty is incomparably better represented in innumerable individuals than in those which can be counted and are finite. God's inaccessible countenance must be reflected in an infinite image in which, as innumerable members, the worlds [meaning the sun and the planets] are as innumerable as the other worlds [meaning different from our Sun and our Earth]. For the same reason that there are innumerable degrees of perfection that must develop the disembodied majesty of God in a bodily manner, there must be innumerable individuals such as these great bodies (of which the Earth is one, our divine mother, which gave birth to us and nurtures us, and which will not try again). Infinite space is needed to hold their quantity without number. Therefore, as it is good that this world exists [the Earth is understood and life on it], that it could be and can be, so too is it good that there really are, could be and can be, innumerable similar worlds to this one [...]
>
> Why should we and can we claim that God's goodness, which can be communicated to an infinite number of things and poured out into infinity, would want to be greedy and withdraw to nothingness, because everything finite is nothing in relation to infinity? Why do you want this center of divinity, which can expand endlessly in the form of an infinite sphere (if it can be so expressed), remain as if unwilling, somewhat sterile, rather than communicating as a creative father, a sublime one and beautiful? Why should it communicate itself to a lesser extent, or to put it better, not communicate at all, rather than being, according to the nature of His magnificent potency, everything? Why should the infinite creative ability be useless, be deprived of the possibility of the existence of innumerable possible worlds? Why should the seriousness of God's image be dimmed, which would have to shine in an unreduced mirror and in an infinite or cosmic way? Why should we claim something that causes such inconsistencies and destroys so many principles of philosophy without in any way benefiting laws, faith, or morals? How do you want God to be limited in potency as well as in activity and effects [...]

Giordano Bruno not only pointed out that a finitely large real world in infinite space would not correspond to God's goodness, majesty and power, but in the following words he also made it clear that it was doubtful about any God who would not manifest himself in an actually infinite work, whether he could manifest himself in this way at all; in other words, whether his creative power was not limited by actual infinity.

> For all those reasons, therefore, that this world of ours, if it is conceived as finite, can be said to be appropriate, good, necessary, all

by Vopěnka. [Translator's note.]

> other innumerable worlds can be said to be appropriate and good; for the same reasons, omnipotence does not deny their existence. And if we don't recognize them, this omnipotence could be accused of not wanting or not being able to make them such, and so leaving a void (or, if you don't want to use the word void, leaving infinite space); this would not only diminish the infinite perfection of being, but also the immeasurable majesty of the effective cause in things created, if created, or in dependent things, if they are eternal. Why should we believe that an agent who can do an infinite amount of good has made it limited? And if he has made it limited, why should we believe that he has the opportunity to make it infinite, when for him the possibility and the realizing activity merge into one?

Giordano Bruno remained true to this conviction until his death. He openly claimed it during his third interrogation before the Inquisition tribunal in 1592, where he said:

> There is an infinite universe that is the result of God's infinite power, for I consider it unworthy of God's goodness and power for the Deity to give birth to one finite world when, in addition to this world, he could give rise to another and infinitely many others. So be it; I have declared that there is an infinite number of individual worlds, similar to this world of our Earth, which I believe with Pythagoras is a moon-like star, other planets, and other stars, of which there are infinitely many, and that all these celestial bodies are innumerable worlds, which then bring together infinite universality in infinite space; and this is called the infinite universe, in which there are innumerable worlds. Thus there are two infinities [he means in actuality, in the real world]: the infinity of the magnitude of the universe and an infinite number of worlds; from which indirectly follows a rejection of truth according to faith.

"Because he remained stubborn not only in this delusion but also in other delusions, on February 16, 1600, servants of justice released him from the dungeon of the Holy Inquisition and took him to the Campo de' Fiori in Rome, where stripping him and tying him to a stake, they burned him alive with his tongue in a vice. Two Dominicans and two Jesuits persuaded him until the last moment to renounce the obstinacy in which he finally completed his miserable unhappy life" – was written in a report by the Brotherhood of the Severed Head of John the Baptist and in other reports from that time. On the same square, three years later, all his books and writings were proclaimed as banned.

1.1.4 Galileo Galilei (1564–1654)

In 1638, one of the last works of the then already famous pioneer of modern European science, Galileo Galilei, was published in the Netherlands. This work is *Discorsi e dimonstrazioni matematiche, intorno a due nuove scienze*, containing four dialogues and a short appendix. The conversations in these dialogues are between Salviati, Sagredo and Simplicio. The first takes Galileo's view and instructs the other two, with Simplicius being less understanding than Sagredo, so Salviati has to explain everything to him a little more broadly.

Although the whole of Galileo's book is remarkably stimulating, what interests us at the moment is only a short excerpt from the first dialogue, included therein really only to illustrate the views expressed there; namely in relation to Simplicio's claim that there are more points on a longer line than on a shorter one. Since what Galileo demonstrates here is now very often explained in a distorted way, let us present this excerpt in a more-or-less literal translation.[5]

> SALV. These are the some of the difficulties that derive from the conversation we have with our finite intellect around infinities, giving them those attributes that we give to things that are finished and ended; which I think is inconvenient, because I believe that these attributes of largeness, smallness and equality do not agree with infinities, of which one cannot be said to be greater or less or equal to the other. As an example I offer a case that has already occurred to me, which for a clearer explanation I put to Mr. Simplicio, who has raised the difficulty. I suppose you know very well which numbers are square, and which are non-squares.
>
> SIMP. I know very well that the square number is the one that arises from the multiplication of another number in itself: and so four, nine, etc., are square numbers, the one being born from the two, and this from the three, multiplied by themselves.
>
> SALV. Very well: and you know further that as the products are called squares, the producers, that is, those that multiply, are called sides or roots; the others, which do not arise from numbers multiplied in themselves, are otherwise not squares. So if I say that all numbers, including squares and non-squares, be more than just squares, I will certainly say a true proposition: isn't it so?
>
> SIMP. We cannot say otherwise.
>
> SALV. If I go on to ask then, how many square numbers there are, I can answer truthfully that there are as many as there are their roots, since it happens that every square has its root, every root its square, nor any square has more than one root, nor any root more than a single square.
>
> SIMP. So it is.
>
> SALV. But if I ask how many roots there are, it cannot be denied that there are not as many as all numbers, since there is no number that is

[5] Galileo Galilei, *Discorsi e dimonstrazioni matematiche, intorno a due nuove scienze* (Leiden: Louis Elsevier, 1638). The English translation presented here is a literal version of the Italian. [Translator's note.]

> not the root of some square; and given this, it will be appropriate to say that square numbers are as many as all numbers, since there are as many as their roots, and roots are all numbers: and even at the beginning we said, all numbers are much more than all squares, being the most not square. And yet the multitude of squares is always decreasing with greater proportion, as more numbers are passed; because up to a hundred there are ten squares, which is the same as saying the tenth part to be squares; in ten thousand only the hundredth part are squares, in one million only the thousandth: and even in the infinite number, if we could conceive it, it should be said, there are as many squares as all the numbers together.
>
> SAGR. What then does this show?
>
> SALV. I do not see that we can come to any other decision than to say that all the numbers are infinite, the squares infinite, their roots infinite, nor the multitude of squares be less than that of all numbers, nor this greater than that, and ultimately, that the attributes of equal, greater, or lesser, make no sense in the infinite, but only in the finite quantities.

Today we would say that the set of all squares of natural numbers can be mapped by a one-to-one mapping onto the set of all natural numbers, and that, consequently, both of these sets have the same **cardinality**. However, Galileo does not mention any such concept. His reasoning is merely meant to be a deterrent to the difficulties and mistakes we might get tangled up in if we talked about infinity using the concepts we have created for finite quantities. He recommends we not talk about the actual infinity at all.

As can be seen, Galileo's advice is unequivocal: Infinity yes, but potential, because in this way we only ever have to actually deal with the finite, and therefore we stand on solid ground. Reflections on actual infinity are dangerous because we do not have the ability to come up with suitable concepts for this infinity, while the concepts created in the study of finite phenomena are not applicable to it.

1.1.5 The Rejection of Actual Infinity

The problem of actual infinity was, from the perspective of the cold intentions of modern European science, at best, a marginal problem. The gaping chasm between man and God, created by the demolition of the Baroque superstructure of the real world, sharply separated science from theology. In this process the problem of actual infinity, thanks to its incomprehensibility by man, fell almost exclusively into the scope of theology. Science saw it as a question impossible to decide, and therefore uninteresting for an active Western European. **Benedict Spinoza** (1632–1677) mentions actual infinity's inaccessibility to either human reason or even human will only as if in passing. He does so in discussing the implications of the forty-ninth proposition in the second volume of his *Ethics* (that is, in examining questions that are more urgent for science):

> If it be said that there is an infinite number of things which we cannot perceive, I answer, that we cannot attain to such things by any thinking, nor, consequently, by any faculty of volition.[6]

Thus, for most of the seventeenth century, influential figures in the nascent modern sciences did not even need to know Galileo's warning words to see clearly not only that potential infinity would suffice for their purposes, but also that considerations of actual infinity unnecessarily, unjustifiably, and dangerously, therefore inadmissibly, exceeded the defined scope of scientific study. For this reason, even Descartes did not pose the question of actual infinity, and **Thomas Hobbes** (1588–1679) wrote in the third chapter of Leviathan:

> Whatsoever we imagine is finite. Therefore there is no idea or conception of anything we call infinite. No man can have in his mind an image of infinite magnitude; nor conceive infinite swiftness, infinite time, or infinite force, or infinite power. When we say anything is infinite, we signify only that we are not able to conceive the ends and bounds of the thing named, having no conception of the thing, but of our own inability. [...] No man therefore can conceive anything, but he must conceive it in some place; and endued with some determinate magnitude; and which may be divided into parts...[7]

These words are, moreover, an eloquent testimony to the origin of the abyss that has opened up in the minds of Western European scholars between man and God. More than by anyone else, its terrifying presence was felt by one of the most famous thinkers of the time, a Catholic but an opponent of the Jesuits, **Blaise Pascal** (1623–1662). In a remarkable document with the apt title *Pensées*, we find the following confession:

> On beholding the blindness and misery of man, on seeing all the universe dumb, and man without light, left to himself, and as it were astray in this corner of the universe, knowing not who has set him here, what he is here for, or what will become of him when he dies, incapable of all knowledge, I begin to be afraid, as a man who has been carried while asleep to a fearful desert island, and who will wake not knowing where he is and without any means of quitting the island.[8]

According to Pascal, man is midway between nothingness represented by zero and infinity, which belongs only to God. There is an insurmountable infinite abyss on both sides, the infinity of which – the same in the direction of greatness as in the direction of insignificance – shows man only its potential form. In his treatise *Of the Geometrical Spirit,* Pascal wrote:

[6] Benedictus de Spinoza, *Ethics,* trans. R. H. M. Elwes (London: G. Bell and Sons, 1887).

[7] Thomas Hobbes, *Hobbes's Leviathan.* Reprinted from the Edition of 1651 (Oxford: Clarendon, 1909), 23.

[8] Blaise Pascal, *Pascal's Pensées; or, Thoughts on Religion*, trans. Gertrude Burford Rawlings (Mount Vernon, N.Y.: Peter Pauper Press, 1900), 7.

> For however quick a movement may be, we can conceive of one still more so; and so on ad infinitum, without ever reaching one that would be swift to such a degree that nothing more could be added to it. And, on the contrary, however slow a movement may be, it can be retarded still more; and thus ad infinitum, without ever reaching such a degree of slowness that we could not thence descend into an infinite number of others, without falling into rest.
>
> In the same manner, however great a number may be, we can conceive of a greater; and thus ad infinitum, without ever reaching one that can no longer be increased. And on the contrary, however small a number may be, as the hundredth or ten thousandth part, we can still conceive of a less; and so on ad infinitum, without ever arriving at zero or nothingness.
>
> However great a space may be, we can conceive of a greater; and thus ad infinitum, without ever arriving at one which can no longer be increased. And, on the contrary, however small a space may be, we can still imagine a smaller; and so on ad infinitum, without ever arriving at one indivisible, which has no longer any extent.
>
> It is the same with time. We can always conceive of a greater without an ultimate, and of a less without arriving at a point and a pure nothingness of duration.
>
> That is, in a word, whatever movement, whatever number, whatever space, whatever time there may be, there is always a greater and a less than these: so that they all stand betwixt nothingness and the infinite, being always infinitely distant from these extremes.[9]

A sharp and open rejection of actual infinity occurred also in the island empire. **John Locke** (1632–1704) went so far in this matter that he did not even grant man the opportunity to create a positive idea of infinity. According to him, the phenomenon of infinity is a mere disposition, that is, a tendency to some action. He writes about how one acquires the idea of infinity in the seventeenth chapter of his *Essay Concerning Human Understanding*:

> Every one that has any idea of any stated lengths of space [...] how often soever he doubles, or any otherwise multiplies it, he finds that after he has continued his doubling in his thoughts, and enlarged his idea as much as he pleases, he has no more reason to stop, nor is one jot nearer the end of such addition, than he was at first setting out. The power of enlarging his idea of space by farther additions remaining still the same, he hence takes the idea of infinite space.[10]

[9] Blaise Pascal, "Of the Geometrical Spirit," trans. Orlando Williams Wight. The Harvard Classics: *Blaise Pascal* (New York: Colier and Son, 1910), 436–37.

[10] John Locke, "An Essay Concerning Human Understanding," in *The Works of John Locke in Nine Volumes,* 12th ed. (London: Rivington, 1824), 1:195.